Random Variables and Cumulative Distribution

A **probability distribution** shows the probabilities observed in an experiment. The quantity observed in a given trial of an experiment is a number called a **random variable (RV)**. In the following, RVs are designated by boldface letters such as **x** and **y**.

- **Discrete RV**: a variable that can only take on certain discrete values.
- **Continuous RV**: a variable that can assume any value within a specified range (possibly infinite).

For a given RV **x**, there are three primary events to consider involving probabilities:

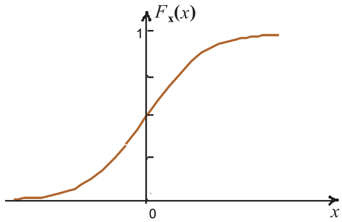
$$\{\mathbf{x} \le a\}, \quad \{a < \mathbf{x} \le b\}, \quad \{\mathbf{x} > b\}$$

For the general event $\{\mathbf{x} \le x\}$, where x is any real number, we define the **cumulative distribution function (CDF)** as

$$F_{\mathbf{x}}(x) = \Pr(\mathbf{x} \le x), \quad -\infty < x < \infty$$

The CDF is a probability and thus satisfies the following properties:

1. $0 \le F_{\mathbf{x}}(x) \le 1, \ -\infty < x < \infty$
2. $F_{\mathbf{x}}(a) \le F_{\mathbf{x}}(b), \ \text{for } a < b$
3. $F_{\mathbf{x}}(-\infty) = 0, \quad F_{\mathbf{x}}(\infty) = 1$



We also note that

$$\Pr(a < \mathbf{x} \le b) = F_{\mathbf{x}}(b) - F_{\mathbf{x}}(a)$$
$$\Pr(\mathbf{x} > x) = 1 - F_{\mathbf{x}}(x)$$

Functions of One RV

In many cases, an examination is necessary of what happens to RV **x** as it passes through various transformations, such as a random signal passing through a nonlinear device. Suppose that the output of some nonlinear device with input **x** can be represented by the new RV:

$$\mathbf{y} = g(\mathbf{x})$$

If the PDF of **x** is known to be $f_{\mathbf{x}}(x)$, and the function $y = g(x)$ has a unique inverse, the PDF of **y** is related by

$$f_{\mathbf{y}}(y) = \frac{f_{\mathbf{x}}(x)}{|g'(x)|}$$

If the inverse of $y = g(x)$ is not unique, and $x_1, x_2, \ldots, x_n$ are all of the values for which $y = g(x_1) = g(x_2) = \cdots = g(x_n)$, then the previous relation is modified to

$$f_{\mathbf{y}}(y) = \frac{f_{\mathbf{x}}(x_1)}{|g'(x_1)|} + \frac{f_{\mathbf{x}}(x_1)}{|g'(x_1)|} + \cdots + \frac{f_{\mathbf{x}}(x_n)}{|g'(x_n)|}$$

Another method for finding the PDF of **y** involves the characteristic function. For example, given that $\mathbf{y} = g(\mathbf{x})$, the characteristic function for **y** can be found directly from the PDF for **x** through the expected value relation

$$\Phi_{\mathbf{y}}(s) = E[e^{isg(\mathbf{x})}] = \int_{-\infty}^{\infty} e^{isg(x)} f_{\mathbf{x}}(x) dx$$

Consequently, the PDF for **y** can be recovered from characteristic function $\Phi_{\mathbf{y}}(s)$ through inverse relation

$$f_{\mathbf{y}}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isy} \Phi_{\mathbf{y}}(s) ds$$

Example: Square-Law Device

The output of a square-law device is defined by the quadratic transformation

$$\mathbf{y} = a\mathbf{x}^2, \quad a > 0$$

where $\mathbf{x}$ is the RV input. Find an expression for the PDF $f_{\mathbf{y}}(y)$ given that we know $f_{\mathbf{x}}(x)$.

Solution: We first observe that if $y < 0$, then $y = ax^2$ has no real solutions; hence, it follows that $f_{\mathbf{y}}(y) = 0$ for $y < 0$.

For $y > 0$, there are two solutions to $y = ax^2$, given by

$$x_1 = \sqrt{\frac{y}{a}}, \quad x_2 = -\sqrt{\frac{y}{a}}$$

where

$$g'(x_1) = 2ax_1 = 2\sqrt{ay}$$
$$g'(x_2) = 2ax_2 = -2\sqrt{ay}$$

In this case, we deduce that the PDF for RV $\mathbf{y}$ is defined by

$$f_{\mathbf{y}}(y) = \frac{1}{2\sqrt{ay}}\left[f_{\mathbf{x}}\left(\sqrt{\frac{y}{a}}\right) + f_{\mathbf{x}}\left(-\sqrt{\frac{y}{a}}\right)\right]U(y)$$

where $U(y)$ is the unit step function.

It can also be shown that the CDF for $\mathbf{y}$ is

$$F_{\mathbf{y}}(y) = \left[F_{\mathbf{x}}\left(\sqrt{\frac{y}{a}}\right) - F_{\mathbf{x}}\left(-\sqrt{\frac{y}{a}}\right)\right]U(y)$$

Example: Correlation and PDF

Consider the random process $\mathbf{x}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$, where ω is a constant and $\mathbf{a}$ and $\mathbf{b}$ are statistically independent Gaussian RVs, satisfying

$$\langle\mathbf{a}\rangle = \langle\mathbf{b}\rangle = 0, \quad \langle\mathbf{a}^2\rangle = \langle\mathbf{b}^2\rangle = \sigma^2$$

Determine

1. the correlation function for $\mathbf{x}(t)$, and
2. the second-order PDF for $\mathbf{x}_1$ and $\mathbf{x}_2$.

Solution: (1) Because $\mathbf{a}$ and $\mathbf{b}$ are statistically independent RVs, it follows that $\langle\mathbf{ab}\rangle = \langle\mathbf{a}\rangle\langle\mathbf{b}\rangle = 0$, and thus

$$\begin{aligned} R_{\mathbf{x}}(t_1,t_2) &= \langle(\mathbf{a}\cos\omega t_1 + \mathbf{b}\sin\omega t_1)(\mathbf{a}\cos\omega t_2 + \mathbf{b}\sin\omega t_2)\rangle \\ &= \langle\mathbf{a}^2\rangle\cos\omega t_1\cos\omega t_2 + \langle\mathbf{b}^2\rangle\sin\omega t_1\sin\omega t_2 \\ &= \sigma^2\cos[\omega(t_2 - t_1)] \end{aligned}$$

or

$$R_{\mathbf{x}}(t_1,t_2) = \sigma^2\cos\omega\tau, \quad \tau = t_2 - t_1$$

(2) The expected value of the random process $\mathbf{x}(t)$ is $\langle\mathbf{x}(t)\rangle = \langle\mathbf{a}\rangle\cos\omega t + \langle\mathbf{b}\rangle\sin\omega t = 0$. Hence, $\sigma_{\mathbf{x}}^2 = R_{\mathbf{x}}(0) = \sigma^2$, and the first-order PDF of $\mathbf{x}(t)$ is given by

$$f_{\mathbf{x}}(x,t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$$

The second-order PDF depends on the correlation coefficient between $\mathbf{x}_1$ and $\mathbf{x}_2$, which, because the mean is zero, can be calculated from

$$\rho_{\mathbf{x}}(\tau) = \frac{R_{\mathbf{x}}(\tau)}{R_{\mathbf{x}}(0)} = \cos\omega\tau$$

and consequently,

$$f_{\mathbf{x}}(x_1,t_1;x_2,t_2) = \frac{1}{2\pi\sigma^2|\sin\omega\tau|}\exp\left(-\frac{x_1^2 - 2x_1x_2\cos\omega\tau + x_2^2}{2\sigma^2\sin^2\omega\tau}\right)$$

Memoryless Nonlinear Transformations

Consider a system in which the output $\mathbf{y}(t_1)$ at time t_1 depends only on the input $\mathbf{x}(t_1)$ and not on any other past or future values of $\mathbf{x}(t)$. If the system is designated by the relation

$$\mathbf{y}(t) = g[\mathbf{x}(t)]$$

where $y = g(x)$ is a function assigning a unique value of y to each value of x, it is said that the system effects a **memoryless** transformation. Because the function $g(x)$ does not depend explicitly on time t, it can also be said that the system is **time invariant**. For example, if $g(x)$ is not a function of time t, it follows that the output of a time invariant system to the input $\mathbf{x}(t+\varepsilon)$ can be expressed as

$$\mathbf{y}(t+\varepsilon) = g[\mathbf{x}(t+\varepsilon)]$$

If input and output are both sampled at times $t_1, t_2, \ldots, t_n$ to produce the samples $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$, respectively, then

$$\mathbf{y}_k = g(\mathbf{x}_k), \quad k = 1, 2, \ldots, n$$

This relation is a **transformation** of the RVs $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ into a new set of RVs $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$. It then follows that the joint density of the RVs $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ can be found directly from the corresponding density of the RVs $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ through the above relationship.

Memoryless processes or fields have no memory of other events in location or time. In probability and statistics, **memorylessness** is a property of certain probability distributions—the exponential distributions of non-negative real numbers and the geometric distributions of non-negative integers. That is, these distributions are derived from Poisson statistics and as such are the only memoryless probability distributions.