Math Fundamentals

PHOE 140 - Intro to Photonics Manufacturing

Abe Michelén
Stonehill College
Easton, Massachussetts

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Engineering is the application of math and physics in order to solve real-world problems. This book provides students with an introduction to the field of engineering, in particular to the disciplines of electrical and computer engineering technology, and review basic mathematical and technical concepts that will prepare students for success in subsequent coursework.

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Dedicated to those students who love learning.

Etre homme, c'est sentir en posand sa pierre que l'on contribue à contruire le monde.

To be a man is to feel that the brick one lays helps to build the world.

— Antoine de Saint-Exupéry, *Terre des Hommnes*

Preface

I have been impressed with the urgency of doing. Knowing is not enough; we must apply. Being willing is not enough; we must do.
— Leonardo da Vinci, *(1452-1519)*

At its heart engineering is about using science to find creative practical solutions. It's a noble profession.
— Queen Elizabeth II, *(*

Course Objectives

1. Be familiar with engineering graphing, drawing, and sketching techniques
2. Be familiar with the unit systems used in engineering, specifically for this course
3. Understand unit dimensional analysis calculations and in checking your final answer for correctness
4. Use exact numbers and significant figures correctly in calculations
5. Conduct linear interpolation on data tables and graphs
6. Obtain a working knowledge of scalars, vectors, and the symbols used in representing them, as related to this course
7. Obtain a working knowledge of voltage, current, power and other electricl parameters
8. Obtain a working knowledge of and be able to solve basic problems related to complex number systems
9. Understand the difference among different types of numbers used in engineering

Part I

Basics

1

Basic Numerical Concepts

. . . all knowledge starts from experience and ends in it. Propositions arrived at by purely logical means are completely empty as regards reality.

— Albert Einstein, *(1879-1955)*

1.1 Objectives

1.2 Theory

In all branches of engineering different types of numbers are used. Integers, decimals, complex numbers, rational, irrational, and other types are very important for different applications. For example electrical engineers use complex numbers to represent such quantities as impedance or phasor[1] In Microcontrollers, Embedded Systems and Programming an integer number is a common term.

This chapter covers the main types used in engineering.

[1] Later in your studies you will recognize these parameters.

1.3 ***Natural and Integers***

These are the type of numbers we normally use for counting things, including negative counts. There are three categories: natural, whole and integers. The difference among them is in the size of the group comprising each category. *Natural num-*

bers start at 1 and end at infinity, e.g., 1, 2, 3, ... *Whole numbers* are all natural numbers including 0, e.g., 0, 1, 2, 3, ... *Integers* include all whole numbers and their negative counterpart e.g., ..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ... A short description of each category follows

1.3.1 *Natural Numbers*

These are the most basic numbers we used in our daily life. Natural numbers **are all positive** numbers, starting with one and going towards infinity: 1, 2, 3, 4, ...

They are also called the **counting numbers** because we use them when we count objects.

We can use natural numbers to:

- count: 1 car, 2 cars, 3 cars
- express our age: 15 years old, 60 years old
- define monetary units: 1 cent, 10 cents, 20 cents

Notice that all these quantities ar strictly positive. Figure 1.1 show the *line number* diagram, where negative, zero (middle tic), and positive numbers are represented in a straight line. The natural numbers are shown to the right of zero

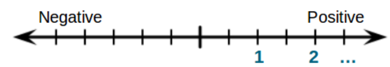

Figure 1.1: Line Numbers Diagram. Natural numbers are the positive numbers

1.3.2 *Whole Numbers*

The **whole numbers system** consists on the natural numbers plus zero (0): 0, 1, 2, 3, ...

Normally we use whole numbers when we need to express a null quantity: no car (0 car).

Figure 1.2 shows the line number diagram. Whole numbers start at the 0 mark.

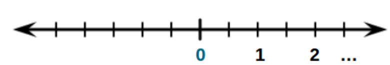

Figure 1.2: Whole numbers are the positive numbers plus 0

1.3.3 Integer Numbers

To represent negative values (a very pervasive task in engineering), the whole numbers are not enough since they are only positive. **Integer numbers** are define as the set of whole numbers plus the set of negative numbers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

Figure 1.3 shows the representation of the integers on the line number diagram.

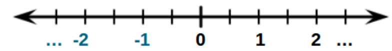

Figure 1.3: Whole numbers are the positive numbers plus 0

Note 1. *We will not strictly define here what are negative numbers. This is the realm of a branch of mathematics called* ***Group Theory.*** *We should,intuitively, say that negative numbers are located at the left of zero.*

We can use integer numbers to represent:

- temperatures: $25°C$, $-25°C$, $-10°F$
- accuracy of a measurement: $\pm 3\%$
- relative voltage: $V_{ab} = -10\ V$

1.4 Fractions and Decimals

This section will study numbers that are a ratio of two numbers (a fraction) and numbers that contain decimals.

1.4.1 Rational Numbers

Any number that can be represented as a ratio (fraction) of two integers is called a **rational number**. The exact definition is given in Note 2

Rational numbers can be expressed as fractions or (if possible) as decimals.

- fractions: $1/2$, $\frac{21}{34}$

Note 2. *A rational number is expresses by the fraction*

$$\frac{x}{y}$$

where x and y are integers, but y (the denominator of the fraction) is non-zero ($y \neq 0$)

Note: Because by dividing an integer by one its value does not change, ***all integers are rational****, but the converse is not true; not all rational numbers are integers.*

- decimals: 0.25, -1.5

Sometimes the result of the fraction consists on an infinity number of decimal places; for example, if we divide 1 by 6 we get $0.16666\ldots$. We express this result as 1.666(6) where 6 in parenthesis means that it is never ending. Also, rational numbers contain all integer numbers, because if the denominator of the fraction is 1 the value of the numbers becomes equal to x (numerator) and this is a integer.

Figure 1.4 represents the line number diagram for rational numbers.

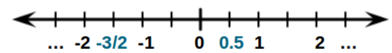

Figure 1.4: Line number diagrams representing rational numb ers.

Note 3. *Even if the numerator or the denominator, or both are not integers, we can always convert them to integers by multiplying both, numerator and denominator, by a suitable power of ten so that both will become integers. For example, the fraction*

$$\frac{10.5}{2.75}$$

is equivalent to

$$\frac{10.5}{2.75} = \frac{(10.5)(1.0 \times 10^2)}{(2.75)(1.0 \times 10^2)} = \frac{1050}{275}$$

which is a rational number.

1.4.2 ***Irrational Numbers***

These type of numbers **cannot be expressed as a ratio of two integers**. Irrational numbers are a special kind of numbers, that generally are solutions of certain equations or are constants used very often in engineering calculations. Examples include,

- π (Greek letter pi): $\pi = 3.141592653589\ldots$

- e, also wn as the **Euler's number** (the base of the natural logarithm): $e = 2.718281828459\ldots$
- some square and cubic roots:
  - the square rot of 2: $\sqrt{2} = 1.414213562373\ldots$
  - the square root of 3: $\sqrt{3} = 1.732050807568\ldots$
- the **Golden Ratio**, $\varphi = 1.6180339887\ldots$ is also irrational (see Note 4).

2.5cm]

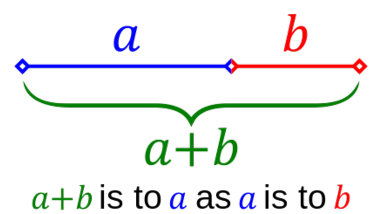

Figure 1.5: Golden Ratio (Wikipedia)

Note 4. *The **Golden Ratio** is the ratio between two quantities (a and b, for instance) that satisfy the following condition: their ratio (a/b) is the same as the ratio of their sum (a + b) to the larger (a) of the two quantities.*
Figure 1.5 shows the geometric relations of this statement. If the Greek letter phi (φ) represents the golden ration, we can express the algebraic relationship as follows:

$$\varphi = \frac{a+b}{a} = \frac{a}{b}$$

If $b = 1$ and $a > b$, then the above equation becomes,

$$a^2 - a - 1 = 0$$

The solution is precisely the Golden Ration, or

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

All these numbers have never ending decimal places and can not be written as the ratio of two integer numbers.

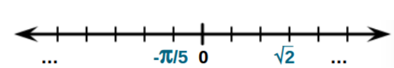

Figure 1.6: Irrational Numbers

Historic Curiosity

Hippasus of Metapontum, (flourished c. 500 BC), philosopher, early follower of Pythagoras, is though that together with Aristotle and Heraclitus identified *fire* as the first element in the universe. Hippasus is also credited with the discovery of the irrational numbers, one of the most important discoveries in the history of science. He found that the side and diagonal of certain figures, such as the square and pentagon, are incommensurable. We know now that if the sides (s) a square are known, the diagonal is given by $d = \sqrt{s^2 + s^2} = \sqrt{2s^2} = s\sqrt{2}$.

It is impossible to measure exactly the value of $\sqrt{2}$, so Hyppasus concluded that $\sqrt{2}$ is a different number than the Pythagorean numbers. He proved that the $\sqrt{2}$ cannot be written as a fraction.

Pythagoras taught his disciples that all numbers could be expressed as the ratio of integers. In other words, all numbers are rational numbers, and the discovery of irrational numbers by Hyppasus is said to have shocked them.

Some traditions say that he was drowned by the gods for divulging the nature of the irrational.

Hyppasus of Metapontum Hippasus of Metapontum (c. 530 – c. 450 BC) was a Pythagorean philosopher.[2] Little is known about his life or his beliefs, but he is sometimes credited with the discovery of the existence of irrational numbers. The discovery of **irrational numbers** is said to have been shocking to the Pythagoreans, and Hippasus is supposed to have drowned at sea, apparently as a punishment from the gods for divulging this. (Wikipedia)

Figure 1.6 shows the line diagram of irrational numbers.

1.4.3 ***Real Numbers***

The combination of the rational and irrational numbers is the set of **real numbers**. So the real numbers include both rational and irrational numbers.

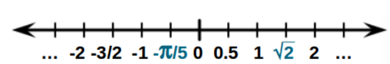

Figure 1.8: Real Numbers

Figure 1.8 shows the number line diagram of real numbers. Any point on the number line is a real number. Since they contain all rational numbers, they also contain natural, whole and integer numbers as well as the irrational numbers.

1.5 ***Complex Numbers***

Complex numbers came about when mathematicians discovered that solutions to some quadratic and cubic equations produced numbers that are non-existent (not real numbers). So the solutions were imaginary or non-real.

All the non-real solutions are related to square roots of negative numbers. Take, for example, the equation

$$x^2 + 9 = 0 \tag{1.51}$$

Its solutions is as follows

$$\begin{aligned} x^2 + 9 &= 0 \\ x^2 &= -9 \\ x &= \sqrt{-9} \\ x &= \sqrt{(9)(-1} \\ x &= 3\sqrt{-1} \end{aligned}$$

but the square root of a negative number (in this case, $\sqrt{-1}$) is not a real number because there is no real number whose square is a negative number (see Note 5). **No real number satisfies this equation**.

To solve this problem, mathematicians defined a new type of number called imaginary number. Its standard symbol is i and it is defined by

$$i = \sqrt{-1} \tag{1.52}$$

Gerolamo Cardano, 1501-1576, Italian, was one of the most influential mathematicians of the Renaissance, and was one of the key figures in the foundation of probability and the earliest introducer of the binomial coefficients and the binomial theorem in the Western world. He conceived and defined complex numbers, who called them "*fictitious*", during his attempts to find solutions to cubic equations. (Wikipedia)

Note 5. *Every number, positive or negative, multiplied by itself will produce* ***all the time*** *a positive number. Which means that the power of two function will only produce positive numbers, or*

$$f(x) = x^2 >= 0$$

This statement tells us that the square root, makes sens only for positive numbers, ***if we only have real numbers!***

Symbol for electrical engineering
The standard symbol, as we said, is the lower-case letter i (for imaginary). However, when using imaginary numbers in electrical or electronic engineering the symbol is replaced with the lower-case j. So,

$$j = \sqrt{-1}$$

Sometimes, the solution of equations results in a combination of real numbers and imaginary numbers. For example, the solution of the quadratic equation

$$x^2 + 2x + 5 = 0 \tag{1.53}$$

is given by

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4-20}}{2} \\ &- \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4i}{2} \end{aligned}$$

therefore Equation 1.53 has two solutions, both involving imaginary numbers,

$$\begin{aligned} x &= -1 + 2i \\ x &= -1 - 2i \end{aligned}$$

We say that these are **complex number** solutions.

Definition: Complex Number
A complex number is a pair of values, one real and one imaginary, expressed as follows

$$z = a + bi$$

or

$$z = a + bj$$

where a and b are real numbers (positive or negative).

The number z is the **complex number**, a is called the **real part** and bi is the **imaginary part**.

Properties and Notation The following properties of complex numbers are important in order to be able to work and solve complex number problems.

Real part. The real part of a complex number can be represented by $a + 0i$, whose imaginary part is 0. The real part of a complex number z is denoted by $Re(z)$. For example given $z = 5 + 3j$, then

$$Re(z) = Re(5 + 3j = 5)$$

Imaginary part. The (purely) imaginary part is given by $0 + bi$. Its real part is 0. The imaginary part is denoted by $Im(z)$. For example, the imaginary part of the previous example is written as

$$Im(z) = Im(5 + 3j) = 3$$

Notation: Cartesian Form. Using the real and imaginary parts, complex numbers are denoted by

Rectangular form:

$$z = x + jy \qquad \text{Rectangular form} \tag{1.54}$$

where x is the real part and y is the imaginary part.

In some cases we can write the imaginary unit $i = j = \sqrt{-1}$ in front of the number representing the imaginary part or at the back. Another way to denote complex numbers is just to list the number representing the real part and the number representing imaginary part as a duple (the same way we represent point in a Cartesian plane) or with and ordered pair enclosed in a parenthesis $(Re(z), Im(z)$. In this notation the complex number $w = c + jd$ is equivalent to $w = (c, d)$. So, the following three expressions are equivalent:

$$z = 2 - 6i$$
$$z = 2 - i6$$
$$z = (2, -6)$$

Representation. Complex numbers can not be represented anymore only on the number line (horizontal) as we have seen before for real numbers, and others. They need two axes for representation because a complex number is represented by two (real) numbers. We have chosen the standard Cartesian plane to visualize complex numbers. This plane is called **Cartesian Complex Plane** when it is used to represent complex numbers. The horizontal axis is used to display the real part, and the vertical axis to represent the imaginary part. In this plane the

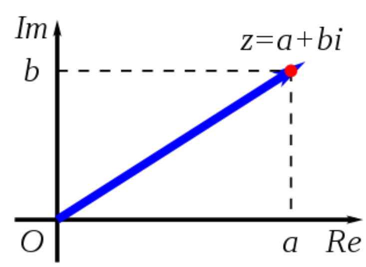

Figure 1.10: A complex number z, as a point (red) and its position vector (blue). (Wikipedia)

number $z = a + jb$ is displayed as the point with coordinates (a,b), and also we indicate the relative size of the number by drawing a vector from the origin of the complex plane to the point (a,b). This distance is called the **position vector** of z. Details are shown in Figure 1.10.

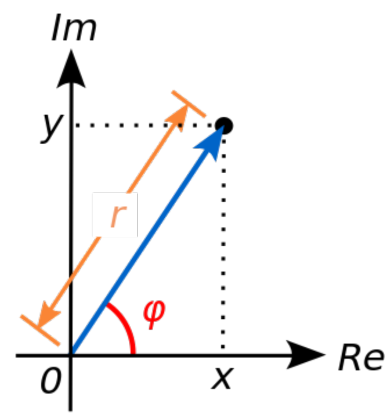

Figure 1.11: Argument φ and modulus r locate a point in the complex plane. (Wikipedia)

Absolute Value and Phase. Similar to absolute values of real numbers, complex numbers do have an **absolute value**. The absolute value of $z = x + yi$ is determined by

$$r = |z| = \sqrt{x^2 + y^2} \tag{1.55}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

The absolute value is also known as **modulus**, the **magnitude**, and the **norm**. Figure 1.11 shows that by Pythagoras' theorem the absolute value of z is the distance from the origin to the point (x, y) representing the complex number.

The **phase** (or the **argument** of a complex number is the angle the position vector makes with the positive real axis. This is shown in Figure 1.11.

The phase is calculated using trigonometry by

$$arg(z) = \varphi = tan^{-1}\left(\frac{y}{x}\right) \tag{1.56}$$

You have to be careful when using Equation 1.56. The value of the angle calculated with equation-depends on the individual values of x and y. In particular, pay attention to the **sign** of each coordinate.

Figure 1.12 shows complex numbers in the first and second quadrants, and Figure 1.13 numbers in the third and fourth quadrants.

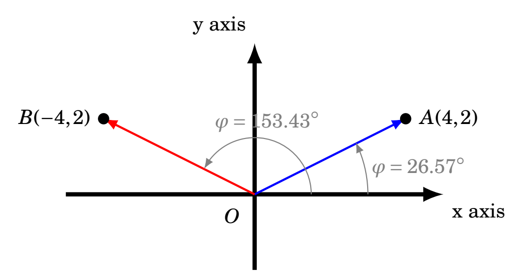

Figure 1.12: First and Second Quadrant

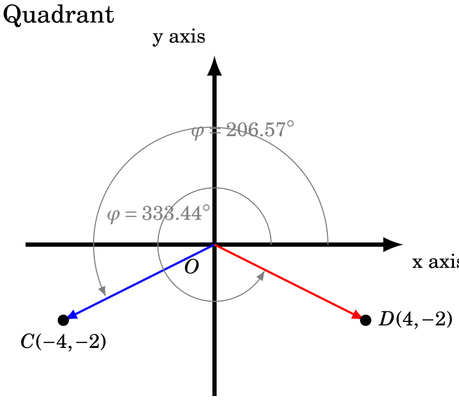

Figure 1.13: Third and Fourth Quadrant

Example 1.5 illustrates these concepts.

Example 1.5.1: Examples Using Eq. 1.56

Some examples using Equation 1.56.

1. $x = 4.0, y = 2.0$. First quadrant (See Figure 1.12)

$$\begin{aligned} \varphi &= tan^{-1}\left(\frac{2.0}{4.0}\right) = 0.5 \\ &= 26.57^{\circ} && first\ quadrant \end{aligned}$$

2. $x = -4.0, y = 2.0$. Second quadrant (See Figure 1.12)

$$\begin{aligned} \varphi &= tan^{-1}\left(\frac{2.0}{-4.0}\right) = -0.5 \\ &= 153.43^{\circ} && Second\ quadrant \end{aligned}$$

3. $x = -4.0, y = -2.0$. Third quadrant (See Figure 1.13)

$$\begin{aligned} \varphi &= tan^{-1}\left(\frac{-2.0}{-4.0}\right) = 0.5 \\ &= 206.57^{\circ} && thirdt\ quadrant \end{aligned}$$

4. $x = 4.0, y = -2.0$. Fourth quadrant (See Figure 1.13)

$$\begin{aligned} \varphi &= tan^{-1}\left(\frac{-4.0}{2.0}\right) = -0.5 \\ &= 333.43^{\circ} && fourtht\ quadrant \end{aligned}$$

Notation: Polar Form. Using the magnitude and phase, we can denote complex number by

Polar form:

$$z = r\underline{/\varphi} \qquad \text{Polar form} \tag{1.57}$$

This form is also called angle form and phasor form. It is used in electronics to represent a phasor with amplitude r and phase φ. The concept of phasors is extremely important in electrical engineering.

where r is the magnitude of z given by Equation (1.55), and φ by Equation (1.56)

Summary: Rectangular and Polar forms. We have seen two forms on how to display complex numbers.

Rectangular form:

$$z = x + jy \qquad \text{Rectangular form} \tag{1.58}$$

Polar form:

$$z = r\underline{/\varphi} \qquad \text{Polar form} \tag{1.59}$$

1.5.1 ***Conversions***

It is important for the students to know how to convert from one form to the other. Sometimes it is more convenient to use Equation (1.58) to solve problems, and in some other instances it is easier to use Equation (1.59). The following paragraphs describe the method used to convert from one from to the other. Refer to Figure 1.11 to understand the conversions.

To convert from rectangular to polar form: given $z = x + jy$:

$$r = |z| = \sqrt{x^2 + y^2} \qquad \text{magnitude} \tag{1.510}$$

$$\varphi = tan^{-1}\left(\frac{y}{y}\right) \qquad \text{argument} \tag{1.511}$$

To convert polar to rectangular form: given $z = r\underline{/\varphi}$:

$$x = r\,cos\,\varphi \qquad \text{real part} \tag{1.512}$$

$$y = r\,sin\,\varphi \qquad \text{imaginary part} \tag{1.513}$$

Examples 1.5.1 and 1.5.1 show how to perform the conversions.

Example 1.5.2: Rectangular to Polar Conversions

Example: Convert the complex number $z = 3 + j4$ to its equivalent polar form.
Solution:

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\varphi = tan^{-1}\left(\frac{4}{3}\right) = 53.13°$$

therefore:

$$z = 5\underline{/53.13°}$$

Example: Given $z = -3 + j4$ determine the equivalent polar form of this complex number.
Solution:

$$r = \sqrt{(-3)^2 + 4^2} = 5$$

$$\varphi = tan^{-1}\left(\frac{4}{-3}\right) = 126.87°$$

(Notice that the point $(-3, 4)$ is in the second quadrant.)

Example 1.5.3: Polar to Rectangular Conversions

Example: Convert the complex number $z = 9\underline{/25^\circ}$ in polar form to its equivalent rectangular form.
Solution:

$$x = 9cos(25^\circ) = 8.16$$
$$y = 9sin(25^\circ) = 3.80$$

therefore:

$$z = 8.16 + j3.80$$

Example: Convert $z = 50\underline{/210^\circ}$ to its rectangular form
Solution:

$$x = 50cos(210^\circ) = -43.30$$
$$y = 50sin(210^\circ) = -25.00$$

Therefore:

$$z = -43.30 - j25$$

(Notice that the point $(-43.30, -25)$ is in the third quadrant.)

1.5.2 *Euler's Formula*

By combining the rectangular-coordinates forms for the real part and imaginary part (Equation (1.512) and (1.513)) with the Cartesian (rectangular) form (Equation (1.58)), we get

$$z = r\left(cos\,\varphi + i\,sin\,\varphi\right) \tag{1.514}$$

This form is called *trigonometric form* and it is related to a formula developed by Leonhard Euler called the *Euler's Formula*, that relates the complex exponential function e^{ix}, where e is the base of the natural logarithm, of a real number x to the sine and cosine of the number. It is expresses as

$$e^{ix} = cos\,x + i\,sin\,x \tag{1.515}$$

We now can write the trigonometirc form of a complex number (Equation (1.514)) using Euler's formula. We get,

$$z = re^{i\varphi} \tag{1.516}$$

This is the *Euler's form* of a complex number.

Leonhard Euler (1707-1783), Swiss. Euler, born in Bern, Switzerland, was a mathematician, physicist, astronomer, geographer, logician and engineer who made important and influential discoveries in many branches of mathematics, such as infinitesimal calculus and graph theory, while also making pioneering contributions to several branches such as topology and analytic number theory. Euler was one of the most eminent mathematicians of the 18th century and is held to be one of the greatest in history. (Wikipedia)

1.5.3 *Summary of Complex Forms*

A summary of the representation of complex numbers follws.

Summary of Complex Forms

Imaginary unit: $i = j = \sqrt{-1}$

Equation forms:

$z = a + jb$	rectangular
$z = r\underline{/\varphi}$	polar, angle or phasor
$z = r\left(cos\,\varphi + j\,sin\,\varphi\right)$	trigonometric
$z = re^{j\varphi}$	$Euler$

Conversions:

$r = \sqrt{a^2 + b^2}$	modulus or magnitude
$\varphi = tan^{-1}\left(\frac{b}{a}\right)$	phase or argument
$a = r\,cos\,\varphi$	real part
$b = r\,sin\,\varphi$	imaginary part

1.6 ***Arithmetic of Complex Numbers***

Complex numbers are *binomials*, so, like real numbers, they are added, subtracted, and multiplied in a similar way. Division, power adn square root are different and we will treat these operations later on.

1.6.1 ***Addition and Subtraction***

The addition or subtraction of complex numbers can be done either mathematically or graphically in rectangular form. For addition, the real parts are firstly added together to form the real part of the sum, and then the imaginary parts to form the imaginary part of the sum. Similarly for subtraction, the real part of the result is the difference of the real parts of the numbers to be subtracted, and the imaginary part of the result is the difference of the two imaginary parts. and this process is as follows using two complex numbers A and B as examples.

Addition and Subtraction

Complex numbers: $Z_1 = a + jb$, $Z_2 = c + jd$

Addition:

$$Z = Z_1 + Z_2 \tag{1.617}$$
$$= (a+c) + j(b+d) \tag{1.618}$$

Subtration:

$$Z = Z_1 - Z_2 \tag{1.619}$$
$$= (a-c) + j(b-d) \tag{1.620}$$

1.6.2 *Powers of* $i = \sqrt{-1}$

Powers of complex numbers in rectangular forms are just special cases of products when the power is a positive whole number. When we perform the operation, however, we must take into consideration the powers of the imaginary unit, $i = \sqrt{-1}$. The following table shows some power of i:

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$
$i^5 = i$	$i^6 = -1$	$i^7 = -i$	$i^8 = 1$
$i^9 = i$	$i^{10} = -1$	$i^{11} = -i$	$i^{12} = 1$

Table 1.1: Powers of $i = \sqrt{-1}$

Notice the pattern: the value of the power of i^n cycles in a period of length 4. For instance, if $i^n = i$ then, also $i^{(4k+n)} = i$, where $k = 0, 1, 2, \ldots$. The same principle applies to all entries in the table. Equation (1.621) shows a summary.

$$i^n = \begin{cases} i & \text{if n is one more than a multiple of 4} \\ -1 & \text{if n is two more than a multiple of 4} \\ -i & \text{if n is three more than a multiple of 4} \\ 1 & \text{if n is a multiple of 4} \end{cases} \tag{1.621}$$

Exercise 1. *Prove that the entries in the previous table (Table 1.1) are correct.*

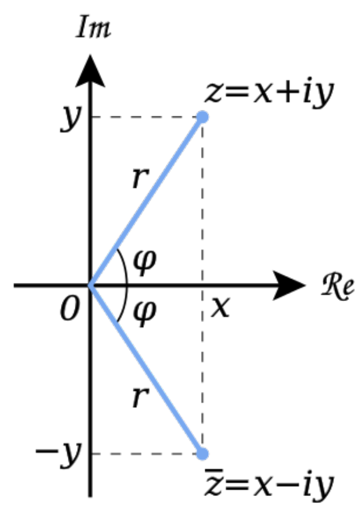

Figure 1.15: Geometric representation (Argand diagram) of z and its conjugate $\bar{z}$ in the complex plane. The complex conjugate is found by **reflecting** z across the real axis. (Wikipedia)

1.6.3 *Complex Conjugate*

The **complex conjugate** of a complex number is a property that is very useful to perform certain applications. The operation of determining the complex conjugate of a complex number consists of changing the sign of the imaginary part of the number. The real part is left unchanged.

Notation: The complex conjugate of a number is displayed by placing a bar on top of the name of the number.

So, in **rectangular form**, if $z = a + jb$, then its conjugate is $\bar{z} = a - jb$. A geometric interpretation is shown in Figure 1.15.

In polar form, the conjugate of $z = re^{i\varphi}$ is $\bar{z} = re^{-i\varphi}$.

Properties of Complex Conjugates

Defnitions:

If $z = a + jb$, then $\overline{z} = a - jb$, rectangular form

If $z = re^{i\varphi}$, then $\overline{z} = re^{-i\varphi}$, polar form

Properties:

- The product of a complex number and its conjugate is a real number.

$$z\overline{z} = a^2 + b^2 = |z|^2 \quad \text{Cartesian form} \tag{1.622}$$

$$z\overline{z} = r^2 \quad \text{Polar form} \tag{1.623}$$

- The inverse of a complex number can be computed using Equation (1.622) above.

$$z\overline{z} = |z|^2 \tag{1.624}$$

$$z = \frac{|z|^2}{\overline{z}} \tag{1.625}$$

$$z^{-1} = \frac{\overline{z}}{|z|^2} \quad \text{cartesian form} \tag{1.626}$$

Conversions:

$$r = \sqrt{a^2 + b^2} \quad \text{modulus or magnitude}$$

$$\varphi = tan^{-1}\left(\frac{b}{a}\right) \quad \text{phase or argument}$$

$$a = r\,cos\,\varphi \quad \text{real part}$$

$$b = r\,sin\,\varphi \quad \text{imaginary part}$$

Exercise 2. *Prove Equations* (1.622) *and* (1.623)

Note 6. ***Special case:*** *Many times we need to determine the value of the inverse of the imaginary unit. We can apply the formula for the inverse of a complex number, as follows:*

$$\begin{aligned} a &= \frac{1}{j} \\ &= \frac{(1)(-j)}{j(-j)} \\ &= \frac{-j}{-j^2} \\ &= \frac{-j}{1} = -j \end{aligned}$$

Therefore:

$$\frac{1}{j} = -j$$

Examples

Example 1.1. *What is the complex conjugate of* *following complex numbers? (a)* $2+3i$*, (b)*$7i$*, (c)* $5-i$

Solutions:

(a) $2-3i$

(b) $-7i$

(c) $5+i$

Example 1.2. *Determine the value of the product of the complex number* $A=4+3i$ *and its conjugate*

Solutions:

$$\begin{aligned} A\overline{A} &= (4+3i)(4-3i) \\ &= (4)(4)+(4)(-3i)+(3i)(4)+(3i)(-3i) \\ &= 16-12i+12i-9i^2 \\ &= 16-9(-1) \\ &= 16+9 \\ &= 25 \end{aligned}$$

This is a proof of property (1.622).

1.6.4 ***Multiplication and Division***

We will divide this section into multiplication and division using rectangular and polar forms.

Multiplication with Rectangular Forms. The **multiplication** of complex numbers in the rectangular form follows more or less the same rules as for normal algebra along with some additional rules for the successive multiplication of the i-operator as described in Table 1.621.

Multiplication: Cartesian Form

Complex numbers: $Z_1 = a + jb$, $Z_2 = c + jd$

Multiplication:

$$\begin{aligned} Z &= Z_1 \times Z_2 \\ &= (a + jb)(c + jd) \\ &= ac + iad + ibc + j^2 bd \\ &= (ac - bd) + j(ad + bc) \end{aligned}$$

Therefore,

$$Re(Z) = ac - bd \qquad \text{Real part of the product} \qquad (1.627)$$

$$Im(Z) = ad + bc \qquad \text{Imagianry part of the product} \qquad (1.628)$$

Division with Rectangular Forms. The division of complex numbers in rectangular form is a little more difficult to perform because the denominator can also be a complex number. A standard procedure is to multiply numerator and denominator by the complex conjugate of the denominator. This procedure will convert the denominator into a pure real number to facilitate the division process. This is called *rationalising*.

A sketch of the process is as follows:

Division: Cartesian Form

Complex numbers: $Z_1 = a + jb$, $Z_2 = c + jd$

Division:

$$\begin{aligned} Z &= \frac{Z_1}{Z_2} \\ &= \frac{Z_1\overline{Z_2}}{Z_2\overline{Z_2}} && \text{multiply by conjugate} \\ &= \frac{Z_1\overline{Z_2}}{|Z_2|^2} && \text{denominator is a real number} \end{aligned}$$

Once the denominator becomes a real number, this operation becomes a standard multiplication of two numbers in Cartesian forms.

Examples

Example 1.3. *Multiply the numbers* $A = 5 + i2$ *and* $B = 2 - i3$

Solutions:

$$\begin{aligned} C &= A \times B \\ &= (5 + i2)(2 - i3) \\ &= 10 - i15 + i4 - i^2(6) \\ &= 10 - i11 - 6(-1) \\ &= 10 + 6 - 11i \\ &= 16 - 11i \end{aligned}$$

Example 1.4. *Divide the numbers* $A = 5 + i2$ *and* $B = 2 - i3$

Solutions:

$$
\begin{aligned}
D &= \frac{A}{B} \\
&= \frac{5+i2}{2-i3} \\
&- \frac{(5+i2)(2+i3)}{(2-i3)(2+i3)} \\
&= \frac{4+i19}{13} \\
&= 0.31+i1.46
\end{aligned}
$$

Multiplication with Polar Forms. Rectangular form is best for adding and subtracting complex numbers as we saw above, but polar form is often better for multiplying and dividing, as we will see here.

To multiply together two vectors or phasors in polar form, we must first multiply together the two modulus or magnitudes and then add together their angles. This is illustrated as follows:

Multiplication: Polar Form

Complex numbers: $Z_1 = r_1\underline{/\varphi_1}$, $Z_2 = r_2\underline{/\varphi_2}$

Multiplication:

$$\begin{aligned} Z &= Z_1 \times Z_2 \\ &= (r_1\underline{/\varphi_1})(r_2\underline{/\varphi_2}) \\ &= (r_1 r_2)\underline{/\varphi_1 + \varphi_2} \qquad \textbf{multiply the magnitudes, add the angles} \end{aligned}$$

Therefore,

$$Z = Z_1 \times Z_2 \tag{1.629}$$

$$= (r_1 r_2)\underline{/\varphi_1 + \varphi_2} \tag{1.630}$$

Examples

Example 1.5. *Multiply the complex numbers* $z = 10\underline{/30^\circ}$ *and* $w = 5\underline{/120^\circ}$ ***Solutions:***

$$\begin{aligned} U &= zw \\ &= (10\underline{/30^\circ})(5\underline{/120^\circ}) \\ &= 50\underline{/150^\circ} \qquad \textit{answer} \end{aligned}$$

Division with Polar Forms. When dividing two vectors or phasors in polar form, we get a polar form number whose magnitude is the ratio of the magnitude of the twop numbers, and its angle is difference between the angle of the numerator and the angle of the denominator. This is illustrated as follows:

Division: Polar Form

Complex numbers: $Z_1 = r_1\angle\varphi_1$, $Z_2 = r_2\angle\varphi_2$

Division:

$$
\begin{aligned}
Z &= \frac{Z_1}{Z_2} \\
&= \frac{r_1\angle\varphi_1}{r_2\angle\varphi_2} \\
&= \frac{r_1}{r_2}\angle\varphi_1 - \varphi_2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Z &= \frac{Z_1}{Z_2} && (1.631) \\
&= \frac{r_1}{r_2}\angle\varphi_1 - \varphi_2 \qquad \text{solution} && (1.632)
\end{aligned}
$$

Examples

Example 1.6. *Divide the complex numbers* $z = 10\angle 30°$ *and* $w = 5\angle 120°$ ***Solutions:***

$$
\begin{aligned}
T &= \frac{z}{w} \\
&= \frac{10\angle 30°}{5\angle 120°} \\
&= 2\angle -90° \qquad \textit{answer}
\end{aligned}
$$

1.7 *Procedure*

2

Engineering Numbers Formats

...what a man means by a term is to be found by observing what he does with it, not by what he says about it.

— Percy W. Bridgman, *(1882-1961)*

Scientific Notation	Engineering Notation
4.59	4.59
4.59×10^3	4.59×10^3
4.59×10^4	45.9×10^3
4.59×10^{11}	459×10^9
4.59×10^{-5}	45.9×10^{-6}

2.1 *Objectives*

2.2 *Theory*

This chapter will provide a brief review of engineering numbers formatted to satisfy certain requirements, and how to use them in practical applications.

2.3 *Significant Figures*

When reporting values that were the result of a measurement or calculated using measured values, it is important to have a way to indicated the certainty of the measurement. The reported values must not appear as more accurate than the the equipment used to make the measurements. This is accomplished through the use of **significant figures**. Significant figures are the digits in a value that are known with some degree of confidence. Consider Table 2.3 that shows measurements using three different instruments[1]. The significant figures of the measurement depend on the accuracy of

[1] Data taken form the Non-destructive Testing (NDT) Resource Center.

the instrument; the more precise is the instrument the more significant figures we can have confidence on.

To work with significant figures when using engineering numbers, we must follow established rules, as is explained nedxt.

Instrtument ⟹	A	B	C
Measured Value	3g	2.53g	2.531g
Precision	1g	0.01g	0.001g
Significant Figures	1	3	4

Table 2.1: Weight Measurements
A⟹ Postage scale
B⟹ Two-pan balance
C⟹ Analytic balance

2.3.1 *Rules for Significant Figures*

There are conventions that must be followed for expressing numbers so that their significant figures are properly indicated. These conventions are:

1. **All non-zero numbers** ARE significant. The number 33.2 has THREE significant figures because all of the digits present are non-zero.

2. **Zeros between two non-zero** digits ARE significant. 2051 has FOUR significant figures. The zero is between a 2 and a 5. 50014 has five significant digits.

3. **Leading zeros** (those to the left of the first non-zero digit) are NOT significant. They are nothing more than "*place holders*." The number 0.54 has only TWO significant figures. 0.000032 also has TWO significant figures. All of the zeros are leading.

4. **Trailing zeros** (the right most zeros) ARE significant *when there is decimal point in the number*(For this reason it is important to give consideration to when a decimal point is used and to keep the trailing zeros to indicate the actual number of significant figures.). There are FOUR significant figures in 92.00 (See Note 7). 400 and 2.00 have three significant figures, whereas 0.050 has two significant figures (the 5 and the 0 to the right of 5)

Note 7. ***Note:*** *92.00 is different from 92; a scientist who measures 92.00 milliliters knows his value to the nearest 1/100th milliliter; meanwhile his colleague who measured 92 milliliters only knows his value to the nearest 1 milliliter. It's important to understand that* ***zero does not mean nothing****. Zero denotes actual information, just like any other number. You cannot tag on zeros that aren't certain to belong there.*

5. Trailing zeros in a whole number **with the decimal shown** ARE significant. Placing a decimal at the end of a number is usually not done. By convention, however, this decimal indicates a significant zero. For example, "540." indicates that the trailing zero IS significant; there are THREE significant figures in this value.
6. Trailing zeros in a whole number **with no decimal shown** are NOT significant. Writing just "540" indicates that the zero is NOT significant, and there are only TWO significant figures in this value. 470,000 has two significant figures; 400 or 4×10^2 indicates only one significant figure. (To indicate that the trailing zeros are significant a decimal point must be added. "400." has three significant digits and is written as 4.00×10^2 in scientific notation.)
7. Exact numbers have an **INFINITE number of significant figures**. *Defined numbers* also have an infinite number of significant digits. For example, 1 meter = 1.00 meters = 1.0000 meters = 1.0000000000000000000 meters, etc. The number of centimeters per inch (2.54) has an infinite number of significant digits, as does the speed of light (299792458 m/s).
8. **For a number in scientific notation**: $N \times 10^x$, all digits comprising N ARE significant by the first 6 rules; "10" and "x" are NOT significant. 5.02×10^4 has THREE significant figures: "5.02." "10 and "4" are not significant. See Note 8 for more details.

Note 8. *Rule 8 provides the opportunity to change the number of significant figures in a value by manipulating its form. For example, let's try writing* 1100 *with THREE significant figures. By rule 6,* 1100 *has TWO significant figures; its two trailing zeros are not significant. If we add a decimal to the end, we have* 1100. *with FOUR significant figures (by rule 5.) But by writing it in* ***scientific notation****:* 1.10×10^3*, we create a THREE-significant-figure value.*

2.3.2 *Rules for Rounding*

When a value contains too many significant figures, it must be rounded off. There are three rules that are commonly used to minimize the error introduced into a value do to rounding.

With the exception of the enumeration of discrete objects (e.g., 5 cars, 113 cents), a measurement is always an approximation. For example, a friend asks you what time it is. You look at your watch and see that it says: 22:44:35. Are you going to tell your friend, "It's 2:44 and 35 seconds"? I suspect not! You'll probably be thoughtful enough to approximate by rounding up to 2:45. If, on the other hand, it's 2:32:14, you might round down to 2:30.

Mathematically, rounding has a more formal definition. The final digit (or significant figure) of any number is actually an approximation.

To round off a number to N significant figures, the following three rules apply:

1. This method involves *underestimating* the value when rounding the five digits 0, 1, 2, 3, and 4. If the digit to the right of the last digit you want to keep (that is, the first digit you want to drop off, N+1) is less than 5, then drop it (and everything to its right.) The value left behind is your rounded value.

2. This method involves *overestimating* the value when rounding the five digits 5, 6, 7, 8, and 9. If the digit in the N+1 place is greater than 5, then drop it (and everything to its right), and raise the last remaining digit by 1.

3. This method takes into account that zero doesn't really require rounding and when rounding 5, this value is exactly centered between the underestimated value if it is rounded down and the overestimated value if it is rounded up. Therefore, **five should be rounded up half of the**

time and down half of the time. Since it would be difficult to keep track of this when performing numerous measurements or calculations, **5 is rounded down when the preceding significant digit is even and 5 is rounded up when the preceding significant digit is odd**. Values less than 5 are rounded down and values greater than 5 are rounded up. For example, 2.785 would be rounded down to 2.78 and 2.775 would be rounded up to 2.78.

In summary: If the digit in the N+1 place is equal to 5, drop it, and if the preceding (Nth) digit is even – leave N alone; if the Nth digit is odd, raise it by 1. This convention is necessary to keep a set of numbers as "balanced" as possible; i.e., if you round down for digits 1-5 (five cases) and up for 6-9 (only 4 cases), the sum of the resulting numbers will be, on average, lower than the sum of the unrounded terms [HUH??]

Examples:

Example 2.1. *Round 4.3127 to four significant figures.*

Solution:

4.3127 has 5 significant figures; we need to drop the final "7". That leaves us 4.312. But following rule 1 above, we note that since 7 is greater than 5, we need to add 1 to the last number we're keeping (the 2.) So our rounded number becomes ***4.313****.*

You can appreciate that 4.313 is a better approximation of 4.3127 than 4.312 would be. Since the "7" we dropped indicates that the true value of the number is "closer" to 4.313 than it is to 4.312, we've created a better approximation by making that change.

Example 2.2. *Round 10.412 to three significant figures.*

Solution:

10.412 has 5 significant figures, so we get rid of the last two. That leaves 10.4. The digit to the right of the "4" is 1 – and 1 is less than 5. So we leave the 4 alone. Our final estimate is ***10.4****.*

Example 2.3. *Round 14.65 to three significant figures.*

Solution:

Here, N+1 = 5. To decide what to do with 6 (our N), recall rule 3. If the number before the 5 is even, we leave it alone when we drop the 5. 6 is even, so our final value is ***14.6****.*

Example 2.4. *Round 1000.3 to four significant figures.*

Solution:

This one's more complex, because it involves zeros. The rules governing whether a digit qualifies as "significant" are more complicated for zeros. Make sure you read the Significant Figures Tutorial before you try to solve this example.

2.3.3 *Significant Digits in Calculations.*

[2] Once the significant figures of numbers used in calculations are determined and used in these calculations, we must deal with the precision and significant figures of the result of calculations. When combining values with different degrees of precision, **the precision of the final answer can be no greater than the least precise measurement**.

[2] Partial support for this work was provided by the NSF-ATE (Advanced Technological Education) program through grant #DUE 0101709. Opinions expressed are those of the authors (NDT Resource C enter) and not necessarily those of the National Science Foundation.

Addition and Subtraction When adding and subtracting, the final answer must be rounded to the same precision (same number of decimal places) as the least precise calculation value, regardless of the significant figures of any one term.

Example: Add $x = 203.221 + 4.7$ and express the result following the rule (see above) for addition.

Solution:

$$x = 203.221 + 4.7 = 207.921$$

The final result, however, should be rounded to only one decimal place, just like the number 4.7.

Therefore

$$x = 207.9$$

Multiplication, Division, and Roots. When multiplying, dividing, or taking roots, the result should have the same number of significant figures as the least precise number in the calculation.

Example 2.3.1: Multiplication, Division, Root

1. **Example:** multiply $y = 8.61 \times 3.1105$ and express the result in the appropriate form.

   Solution:

   $$y = (8.61)(3.1105) = 26.781405$$

   The final answer should be rounded to two decimal places: $y = 26.78$

2. **Example:** Divide $t = \frac{2.1536}{1.8}$

   Solution:

   $$t = \frac{2.1536}{1.8} = 1.1988444\ldots$$

   The final asnwer must be rounded to two significant figures (due to 1.8): $t = 1.2$

3. **Example 2.5.** *Determine* $s = \sqrt{2.41}$.

   $$s = \sqrt{2.41} = 1.552417\ldots$$

   Then, $s = 1.55$

Logarithms and Antilogithms. When calculating the **logarithm of a number**, the **mantissa** (the number to the right of the decimal point in the logarithm) must have **the same number of significant figures as there are in the number whose logarithm is being found**.

Example 2.3.2: Logarithm

Example: Determine the logarithm (base-10) of $w = log(3.000 \times 10^4)$and express the result in the appropriate form.
Solution:

$$w = log(3.000 \times 10^4 = 4.4771212\ldots$$

Now, the number of significant figures of 3.000 is 4, so the **final answer should have 4 decimal places**.Result: $w = 4.4771$

When calculating the **antilogarithm of a number**, the result should have the same number of significant figures **as the mantissa in the logarithm**.

Example 2.3.3: Antilogarithm

- **Example:** Determine the antilogarithm (base-10) of $v = antilog(0.502))$ and express the result in the appropriate form.

  Solution:

  $$v = antilog(0.502) = 10^{0.502} = 3.17687\ldots$$

  The number of significant figures of the mantissa of the log is 3, so the **final answer should have 3 significant figures**. Result: $v = 3.18$
  Example: Determine the antilogarithm (base-10) of $u = antilog(0.5))$ and express the result in the appropriate form.

  Solution:

  $$u = antilog(0.50) = 10^{0.5} = 3.162277\ldots\ldots$$

  The number of significant figures of the mantissa of the log is 2, so the **final answer should have 2 significant figures**. Result: $u = 3.2$

Multiple Calculations. If a calculation involves a combination of mathematical operations,

- perform the calculation and express the result using more figures than will be significant to arrive at a value.

- go back and look at the individual steps of the calculation and determine how many significant figures would carry through to the final result based on the above conventions.

An example:

Example 2.3.4: Combination

Example: Determine the value of

$$z = \frac{5.254 + 0.0016}{34.6} - 2.231 \times 10^{-3}$$

Solution: After preforming the operations (not taking into account the significant figures of each number) we get

$$z = 0.1496649538$$

Now, let's look at each calculation in the equation in order to determine the final value:

$5.254 + 0.0016 = 5.25566$ The result of this operation shuold have only 3 decimals because the sum should have the same number of decimals as the number with the least number of decimals (5.254), then the result is: $\mathbf{5.254 + 0.0016 = 5.256}$

$\frac{5.256}{34.6} = 0.1519075\ldots$ The result should have the same number of significant figures as the number with the least significant figures (34.6). Therefore $\mathbf{\frac{2.56}{34.6} = 0.152}$

$0.152 - 0.002231 = 0.149769$ The final result of this operation should only have 3 decimals. So, $\mathbf{0.152 - 0.002231 = 0.149}$

Therefore the value obtained ($z = 0.1496649538$) should be rounded to three decimal places. or,

$$\mathbf{z = 0.150 = 1.50 \times 10^{-1}}$$

2.3.4 *Percent Error*

Many times we have to compare a measured value (approximate value) to the know value of that measurement(exact value). When reporting the measured value we must estimate the the difference between the two values. Normally we represent this difference int terms of a percentage. This is called

percent error or **percentage error**.

The purpose of a percent error calculation is to determine *how close* a measured value is to a true value. In some fields, percent error is always expressed as a positive number. In others, it is correct to have either a positive or negative value. The sign may be kept to determine whether recorded values consistently fall above or below expected values.

Percent error is related to two error calculation types: **absolute error** and **relative error**. Absolute error is the difference between an experimental and known valu. When you divide that number by the known value you get relative error. Percent error is relative error multiplied by 100%.

To calculate percent error, use Equation 2.3.4

$$\%_{error} = \left| \frac{\#_{measured} - \#_{exact}}{\#_{exact}} \right| \times 100 \qquad (2.31)$$

Note 9. *A percentage very close to zero means you are very close to your targeted value, which is good. It is always necessary to understand the cause of the error, such as whether it is due to the imprecision of your equipment, your own estimations, or a mistake in your experiment. See Example 2.3.4*

An example:

Example 2.3.5: Speed of Light

Example: The 17th century Danish astronomer, Ole Rømer, observed that the periods of the satellites of Jupiter would appear to fluctuate depending on the distance of Jupiter from Earth. The further away Jupiter was, the longer the satellites would take to appear from behind the planet. In 1676, he determined that this phenomenon was due to the fact that the speed of light was finite, and subsequently estimated its velocity to be approximately 220,000 km/s. The current accepted value of the speed of light is almost 299,800 km/s. What was the percent error of Rømer's estimate?

Solution:

Experimental value $=220,000\ km/s = 2.2 \times 10^8\ m/s$

theoretical value $= 299,800\ km/s = 2.998 \times\ 10^8\ m/s$

Then

$$\%_{error} = \left| \frac{2.2 \times 10^8\ m/s - 2.998 \times 10^8\ m/s}{2.998 \times 10^8\ m/s} \right| \times 100 = 26.62\%$$

So Rømer was quite a bit off by our standards today, but considering he came up with this estimate at a time when a majority of respected astronomers, like Cassini, still believed that the speed of light was infinite, his conclusion was an outstanding contribution to the field of astronomy.

This example was taken form the Department of Physics and Astronomy of the University of Iowa (http://astro.physics.uiowa.edu/)

2.3.5 Exponents

2.3.6 Engineering and Scientific Notations

2.3.7 System of Units and Conversions

2.3.8 Dimensional Analysis

2.4 Procedure

3
Math Concepts

Mathematics reveals its secrets only to those who approach it with pure love, for its own beauty.

— **Archimedesi,** *287 BC - 212 BC*

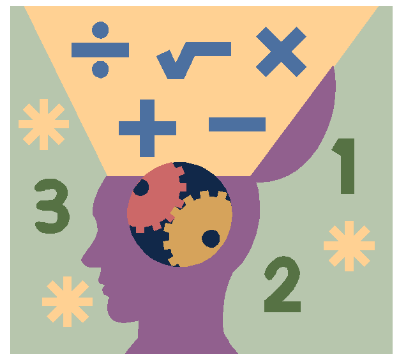

Figure 3.1: Math Concepts

3.1 Objectives

3.2 Theory

3.2.1